

A construction of pro- C^* -algebras from pro- C^* -correspondences

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Abstract

We associate a pro- C^* -algebra to a pro- C^* -correspondence and show that this construction generalizes the construction of crossed products by Hilbert pro- C^* -bimodules and the construction of pro- C^* -crossed products by strong bounded automorphisms.

This paper is dedicated to the memory of Anastasios Mallios

1 Introduction

The notion of a Hilbert C^* -module has first been introduced by I. Kaplansky in 1953. It is a generalization of a Hilbert space, in the sense that the inner product in a Hilbert C^* -module takes values in a C^* -algebra. Since 1953, there has been a continuous development of the theory of Hilbert C^* -modules which has offered a very rich literature and useful tools for various important fields of mathematics, such as KK-theory, C^* -algebraic quantum group theory and groupoid C^* -algebras.

A C^* -correspondence is a natural generalization of a Hilbert C^* -bimodule. Namely it is a pair (X, A) , where X is a right Hilbert A -module together with a left action of A on X . In [Pi], M.V. Pimsner first showed how to associate a C^* -algebra to certain C^* -correspondences, introducing a class of C^* -algebras that are now known as Cuntz-Pimsner algebras. It was later that T. Katsura, in his series of papers [K1, K2, K3], extended the former construction and associated a certain C^* -algebra to every C^* -correspondence. Katsura's more general construction includes a wide range of algebras, amongst them the crossed product of a C^* -algebra by a Hilbert C^* -bimodule, which was introduced in [AEE].

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The extension of so rich in results concepts to the case of pro- C^* -algebras could not be disregarded. A pro- C^* -algebra $A[\tau_\Gamma]$ is a complete topological $*$ -algebra for which there exists a directed family of C^* -seminorms $\Gamma = \{p_\lambda : \lambda \in \Lambda\}$ defining the topology τ_Γ . In 1988, N.C. Phillips considered Hilbert modules over pro- C^* -algebras and studied their structure, in [Ph]. An extensive survey of the theory of Hilbert modules over pro- C^* -algebras can be found in [J1]. In [Z] the notion of a Hilbert pro- C^* -bimodule over a pro- C^* -algebra was defined. Subsequently, in [JZ] we defined and studied the crossed product of a pro- C^* -algebra by a Hilbert pro- C^* -bimodule, which is a generalization of crossed products of pro- C^* -algebras by inverse limit automorphisms (for the latter see [J2]). All the above, gave us the impetus to generalize the important topic of C^* -correspondences in the setting of pro- C^* -algebras and to examine under which conditions we can associate a pro- C^* -algebra to a pro- C^* -correspondence (for the latter see Definition 3.1).

The paper is organized as follows. In Section 2 we gather some basic facts on pro- C^* -algebras and Hilbert pro- C^* -modules that are needed for understanding the main results of this paper. Sections 3 and 4 are devoted in the definition of pro- C^* -correspondences and representations of them respectively. In Section 5 we prove that for a certain pro- C^* -correspondence, namely an inverse limit pro- C^* -correspondence as we shall call it, a universal pro- C^* -algebra can be associated to it, and in Section 6, we see that in case X is a Hilbert pro- C^* -bimodule over a pro- C^* -algebra A the crossed product of A by X is isomorphic to the pro- C^* -algebra associated to X , when the latter is regarded as a pro- C^* -correspondence. Finally, in Section 7, as an application, we show how the association of a pro- C^* -algebra to a pro- C^* -correspondence described in Section 5, generalizes the construction of the crossed product of a pro- C^* -algebra by a strong bounded automorphism.

2 Preliminaries

All vector spaces and algebras we deal with are considered over the field \mathbb{C} of complex numbers and all topological spaces are assumed Hausdorff.

A *pro- C^* algebra* $A[\tau_\Gamma]$ is a complete topological $*$ -algebra for which there exists an upward directed family Γ of C^* -seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$ defining the topology τ_Γ ([F, Definition 7.5]). Other terms with which pro- C^* -algebras can be found in the literature are: locally C^* -algebras (A. Inoue), b^* -algebras (C. Apostol) and LMC*-algebras (G. Lassner, K. Schmüdgen).

For a pro- C^* -algebra $A[\tau_\Gamma]$ and for every $\lambda \in \Lambda$, the quotient normed $*$ -algebra $A_\lambda = A/N_\lambda$, where $N_\lambda = \{a \in A : p_\lambda(a) = 0\}$, is already complete, hence a C^* -algebra in the norm $\|a + N_\lambda\|_{A_\lambda} = p_\lambda(a)$, $a \in A$ ([F, Theorem 10.24]). The canonical map from A to A_λ is denoted by π_λ^A . For $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a canonical surjective C^* -morphism $\pi_{\lambda\mu}^A : A_\lambda \rightarrow A_\mu$, such that $\pi_{\lambda\mu}^A(a + N_\lambda) = a + N_\mu$ for all $a \in A$. The Arens-Michael decomposition gives us the representation of A as an inverse limit of C^* -algebras, namely $A = \varprojlim_{\lambda} A_\lambda$, up to a topological $*$ -isomorphism ([F, p. 15-16]). We refer the reader to [F] for

further information about pro- C^* -algebras.

Given two pro- C^* -algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$, a continuous $*$ -morphism $\varphi : A \rightarrow B$ is called a *pro- C^* -morphism*.

Here we recall some basic facts from [J1] and [Z] regarding Hilbert pro- C^* -modules and Hilbert pro- C^* -bimodules, respectively.

Let $A[\tau_\Gamma]$ be a pro- C^* -algebra. A *right Hilbert pro- C^* -module X over A* (or just *Hilbert A -module*), is a linear space X that is also a right A -module equipped with a right A -valued inner product $\langle \cdot, \cdot \rangle_A$, that is \mathbb{C} - and A -linear in the second variable and conjugate linear in the first variable, with the following properties:

1. $\langle x, x \rangle_A \geq 0$, $\forall x \in X$, and $\langle x, x \rangle_A = 0$ if and only if $x = 0$,
2. $(\langle x, y \rangle_A)^* = \langle y, x \rangle_A$, $\forall x, y \in X$,

and which is complete with respect to the topology given by the family of seminorms $\{p_\lambda^A\}_{\lambda \in \Lambda}$, with $p_\lambda^A(x) = p_\lambda(\langle x, x \rangle_A)^{\frac{1}{2}}$, $x \in X$.

A Hilbert A -module is *full* if the pro- C^* -subalgebra of A generated by $\{\langle x, y \rangle_A; x, y \in X\}$ coincides with A .

A *left Hilbert pro- C^* -module X* over a pro- C^* -algebra $A[\tau_\Gamma]$ is defined in the same way, where for instance the completeness is requested with respect to the family of seminorms $\{^A p_\lambda\}_{\lambda \in \Lambda}$, where $^A p_\lambda(x) = p_\lambda(^A \langle x, x \rangle)^{\frac{1}{2}}$, $x \in X$.

In case X is a left Hilbert pro- C^* -module over $A[\tau_\Gamma]$ and a right Hilbert pro- C^* -module over $B[\tau_{\Gamma'}]$ ($\tau_{\Gamma'}$ is given by the family of C^* -seminorms $\{q_\lambda\}_{\lambda \in \Lambda}$), such that the following relations hold:

- $^A \langle x, y \rangle z = x \langle y, z \rangle_B$ for all $x, y, z \in X$,
- $q_\lambda^B(ax) \leq p_\lambda(a)q_\lambda^B(x)$ and $^A p_\lambda(xb) \leq q_\lambda(b)^A p_\lambda(x)$, for all $x \in X$, $a \in A$, $b \in B$ and for all $\lambda \in \Lambda$,

then we say that X is a *Hilbert $A - B$ pro- C^* -bimodule*.

A Hilbert $A - B$ pro- C^* -bimodule X is *full* if it is full as a right and as a left Hilbert pro- C^* -module.

Let Λ be an upward directed set and $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}; \chi_{\lambda\mu}; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ an inverse system of Hilbert C^* -bimodules, that is:

- $\{A_\lambda; \pi_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ and $\{B_\lambda; \chi_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ are inverse systems of C^* -algebras;
- $\{X_\lambda; \sigma_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of Banach spaces;
- for each $\lambda \in \Lambda$, X_λ is a Hilbert $A_\lambda - B_\lambda$ C^* -bimodule;
- $\langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y) \rangle_{B_\mu} = \chi_{\lambda\mu}(\langle x, y \rangle_{B_\lambda})$ and $_{A_\mu} \langle \sigma_{\lambda\mu}(x), \sigma_{\lambda\mu}(y) \rangle = \pi_{\lambda\mu}(_{A_\lambda} \langle x, y \rangle)$, for all $x, y \in X_\lambda$ and for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$;
- $\sigma_{\lambda\mu}(x)\chi_{\lambda\mu}(b) = \sigma_{\lambda\mu}(xb)$, $\pi_{\lambda\mu}(a)\sigma_{\lambda\mu}(x) = \sigma_{\lambda\mu}(ax)$, for all $x \in X_\lambda$, $a \in A_\lambda$, $b \in B_\lambda$ and for all $\lambda, \mu \in \Lambda$ such that $\lambda \geq \mu$.

Let $A = \varprojlim_{\lambda} A_\lambda$, $B = \varprojlim_{\lambda} B_\lambda$ and $X = \varprojlim_{\lambda} X_\lambda$. Then X has the structure of a Hilbert $A - B$ pro- C^* -bimodule with

$$(x_\lambda)_{\lambda \in \Lambda} (b_\lambda)_{\lambda \in \Lambda} = (x_\lambda b_\lambda)_{\lambda \in \Lambda} \text{ and } \langle (x_\lambda)_{\lambda \in \Lambda}, (y_\lambda)_{\lambda \in \Lambda} \rangle_B = (\langle x_\lambda, y_\lambda \rangle_{B_\lambda})_{\lambda \in \Lambda} \\ \text{and} \\ (a_\lambda)_{\lambda \in \Lambda} (x_\lambda)_{\lambda \in \Lambda} = (a_\lambda x_\lambda)_{\lambda \in \Lambda} \text{ and } {}_A \langle (x_\lambda)_{\lambda \in \Lambda}, (y_\lambda)_{\lambda \in \Lambda} \rangle = ({}_A \langle x_\lambda, y_\lambda \rangle)_{\lambda \in \Lambda},$$

where $(x_\lambda)_{\lambda \in \Lambda} \in X$, $(b_\lambda)_{\lambda \in \Lambda} \in B$ and $(a_\lambda)_{\lambda \in \Lambda} \in A$.

Let X be a Hilbert $A - B$ pro- C^* -bimodule. Then, for each $\lambda \in \Lambda$, ${}^A p_\lambda(x) = q_\lambda^B(x)$, for all $x \in X$, and the normed space $X_\lambda = X/N_\lambda^B$, where $N_\lambda^B = \{x \in X; q_\lambda^B(x) = 0\}$, is complete in the norm $\|x + N_\lambda^B\|_{X_\lambda} = q_\lambda^B(x)$, $x \in X$. Moreover, X_λ has a canonical structure of a Hilbert $A_\lambda - B_\lambda$ C^* -bimodule with $\langle x + N_\lambda^B, y + N_\lambda^B \rangle_{B_\lambda} = \langle x, y \rangle_B + \ker q_\lambda$ and ${}_A \langle x + N_\lambda^B, y + N_\lambda^B \rangle = {}_A \langle x, y \rangle + \ker p_\lambda$, for all $x, y \in X$. The canonical surjection from X on X_λ is denoted by σ_λ^X . For $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a canonical surjective linear map $\sigma_{\lambda\mu}^X : X_\lambda \rightarrow X_\mu$ such that $\sigma_{\lambda\mu}^X(x + N_\lambda^B) = x + N_\mu^B$ for all $x \in X$. Then $\{A_\lambda; B_\lambda; X_\lambda; \pi_{\lambda\mu}^A; \sigma_{\lambda\mu}^X; \pi_{\lambda\mu}^B; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of Hilbert C^* -bimodules in the above sense.

Let X be a Hilbert pro- C^* -module over B . A morphism $T : X \rightarrow X$ of right modules is *adjointable* if there is another morphism of modules $T^* : X \rightarrow X$ such that $\langle Tx_1, x_2 \rangle_B = \langle x_1, T^*x_2 \rangle_B$ for all $x_1, x_2 \in X$. The vector space $L_B(X)$ of all adjointable module morphisms from X to X has a structure of a pro- C^* -algebra under the topology given by the family of C^* -seminorms $\{q_{\lambda, L_B(X)}\}_{\lambda \in \Lambda}$, where

$$q_{\lambda, L_B(X)}(T) = \sup\{q_\lambda^B(Tx) : q_\lambda^B(x) \leq 1\}, \forall \lambda \in \Lambda, T \in L_B(X).$$

Moreover, $\{L_{B_\lambda}(X_\lambda); \pi_{\lambda\mu}^{L_B(X)}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ where $\pi_{\lambda\mu}^{L_B(X)} : L_{B_\lambda}(X_\lambda) \rightarrow L_{B_\mu}(X_\mu)$ is given by $\pi_{\lambda\mu}^{L_B(X)}(T)(\sigma_\mu^X(x)) = \sigma_{\lambda\mu}^X(T(\sigma_\lambda^X(x)))$, for all $T \in L_B(X)$, $x \in X$, is an inverse system of C^* -algebras and $L_B(X) = \varprojlim_{\lambda} L_{B_\lambda}(X_\lambda)$, up to an isomorphism of pro- C^* -algebras. The canonical projections $\pi_\lambda^{L_B(X)} : L_B(X) \rightarrow L_{B_\lambda}(X_\lambda)$, $\lambda \in \Lambda$, are given by $\pi_\lambda^{L_B(X)}(T)(\sigma_\lambda^X(x)) = \sigma_\lambda^X(T(x))$ for all $T \in L_B(X)$ and $x \in X$. For $x, y \in X$, the map

$$\theta_{y,x} : X \rightarrow X, \text{ given by } \theta_{y,x}(z) = y \langle x, z \rangle_B, \forall x, y, z \in X,$$

is an adjointable module morphism. $\Theta(X) := \text{span}\{\theta_{y,x} : x, y \in X\}$ is a two-sided $*$ -ideal of $L_B(X)$ and its closure in $L_B(X)$ is denoted by $K_B(X)$. Moreover, $(K_B(X))_\lambda = K_{B_\lambda}(X_\lambda)$, for each $\lambda \in \Lambda$, with respect to an isomorphism of C^* -algebras.

Throughout this paper, A and B are pro- C^* -algebras whose topologies are given by the families of C^* -seminorms $\{p_\lambda, \lambda \in \Lambda\}$, respectively $\{q_\delta, \delta \in \Delta\}$.

3 Pro- C^* -correspondences

Definition 3.1. A *pro- C^* -correspondence* is a triple (X, A, φ_X) , where A is a pro- C^* -algebra, X is a Hilbert pro- C^* -module over A and $\varphi_X : A \rightarrow L_A(X)$ is

a pro- C^* -morphism.

A pro- C^* -correspondence (X, A, φ_X) is *nondegenerate* if φ_X is nondegenerate (that is, $[\varphi_X(A)X] = X$, where $[\varphi_X(A)X]$ stands for the closure of the linear span of the set $\{\varphi_X(a)x : a \in A, x \in X\}$).

Example 3.2. Let A be a pro- C^* -algebra and $\alpha : A \rightarrow A$ a nondegenerate pro- C^* -morphism. Consider $\varphi_A : A \rightarrow L_A(A)$ defined by $\varphi_A(a)(b) = \alpha(a)b$, $a, b \in A$. Clearly, φ_A is a pro- C^* -morphism and $[\varphi_A(A)A] = A$. Therefore, (A, A, φ_A) is a nondegenerate pro- C^* -correspondence. If $\alpha = \text{id}_A$, we say that (A, A, id_A) is the *identity pro- C^* -correspondence*.

Example 3.3. Suppose that X is a Hilbert $A - A$ pro- C^* -bimodule. Then the map $\varphi_X : A \rightarrow L_A(X)$ defined by $\varphi_X(a)(x) = ax$, $a \in A, x \in X$, is a pro- C^* -morphism and since $[AX] = X$, (X, A, φ_X) is a nondegenerate pro- C^* -correspondence.

Example 3.4. Suppose that (X, A, φ_X) and (Y, A, φ_Y) are pro- C^* -correspondences. By [J1, pp.77-79], $X \otimes_{\varphi_Y} Y$ is a Hilbert pro- C^* -module over A and the map $\varphi_{X \otimes_{\varphi_Y} Y} : A \rightarrow L_A(X \otimes_{\varphi_Y} Y)$ defined by

$$\varphi_{X \otimes_{\varphi_Y} Y}(a)(x \otimes_{\varphi_Y} y) = \varphi_X(a)(x) \otimes_{\varphi_Y} y, \quad a \in A, x \in X, y \in Y,$$

is a pro- C^* -morphism [J1, Proposition 4.3.4]. Then $(X \otimes_{\varphi_Y} Y, A, \varphi_{X \otimes_{\varphi_Y} Y})$ is a pro- C^* -correspondence called the tensor product of the pro- C^* -correspondences (X, A, φ_X) and (Y, A, φ_Y) .

Definition 3.5. A pro- C^* -correspondence (X, A, φ_X) is an *inverse limit pro- C^* -correspondence*, if A is an inverse limit, $\lim_{\leftarrow \lambda} A_\lambda$, of C^* -algebras in such a way that X is an inverse limit, $\lim_{\leftarrow \lambda} X_\lambda$, of Hilbert C^* -modules, where X_λ is a Hilbert A_λ -module for each λ and φ_X is an inverse limit, $\lim_{\leftarrow \lambda} \varphi_{X_\lambda}$, of C^* -morphisms.

Example 3.6. The identity pro- C^* -correspondence and the Hilbert pro- C^* -bimodules are inverse limit pro- C^* -correspondences.

Throughout this paper an ideal of a pro- C^* -algebra always means a closed two-sided $*$ -ideal. For a pro- C^* -correspondence (X, A, φ_X) and an ideal I of A , the following ideals of A are defined (see [K3, Definition 4.1]):

$$\begin{aligned} X(I) &= \overline{\text{span}}\{ \langle y, \varphi_X(a)x \rangle_A : a \in I, x, y \in X \}, \\ X^{-1}(I) &= \{ a \in A : \langle y, \varphi_X(a)x \rangle_A \in I, \forall x, y \in X \}. \end{aligned}$$

Lemma 3.7. Let X be a Hilbert A -module and I an ideal of A . We put $XI = \text{span}\{xa : x \in X, a \in I\}$. Then $x \in XI$ if and only if $\langle y, x \rangle_A \in I$, for all $y \in X$.

Proof. The forward implication is immediate. For the inverse, we have that $\langle x, x \rangle_A \in I$, hence from [J1, Corollary 1.3.11], if α is a real number, $0 < \alpha < \frac{1}{2}$, then there exists $y \in X$, such that $x = y \langle x, x \rangle_A^\alpha$. From functional calculus in pro- C^* -algebras (see [F]), we then have that $\langle x, x \rangle_A^\alpha \in I$, so $x \in XI$. \square

Based on the previous lemma, we get that XI is a closed submodule of X . In particular if $I = \ker p_\lambda$, then by a proof similar to that of Lemma 3.7, we have that $\ker p_\lambda^A = X \ker p_\lambda$, so $X/X \ker p_\lambda = X_\lambda$.

Remark 3.8. If by ϕ_I we denote the $*$ -morphism $\phi_I : L_A(X) \rightarrow L_A(X/XI)$ given by :

$$\phi_I(T)(x + XI) = Tx + XI, T \in L_A(X), x \in X,$$

then we get that $X^{-1}(I) = \ker(\phi_I \circ \varphi_X)$. In particular, if $I = \ker p_\lambda$, then $X^{-1}(\ker p_\lambda) = \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)$.

Lemma 3.9. *A pro- C^* -correspondence (X, A, φ_X) is an inverse limit pro- C^* -correspondence if and only if $X(\ker p_\lambda) \subset \ker p_\lambda$, for all $\lambda \in \Lambda$.*

Proof. Suppose that (X, A, φ_X) is an inverse limit pro- C^* -correspondence. Then $\varphi_X = \lim_{\leftarrow \lambda} \varphi_{X_\lambda}$. Let $\langle y, \varphi_X(a)x \rangle_A \in X(\ker p_\lambda)$, for $x, y \in X, a \in \ker p_\lambda$. Then

$$\begin{aligned} \pi_\lambda^A(\langle y, \varphi_X(a)x \rangle_A) &= \langle \sigma_\lambda^X(y), \pi_\lambda^{L_A(X)}(\varphi_X(a)) \sigma_\lambda^X(x) \rangle_{A_\lambda} \\ &= \langle \sigma_\lambda^X(y), \varphi_{X_\lambda}(\pi_\lambda^A(a)) \sigma_\lambda^X(x) \rangle_{A_\lambda} = 0, \end{aligned}$$

and so $\langle y, \varphi_X(a)x \rangle_A \in \ker p_\lambda$.

Conversely, let $\lambda \in \Lambda$. If $a \in \ker p_\lambda$, then $\langle y, \varphi_X(a)x \rangle_A \in X(\ker p_\lambda) \subset \ker p_\lambda$, for all $x, y \in X$, whence $\varphi_X(a)x \in \ker p_\lambda^A$, for all $x \in X$. Also, since $\ker p_\lambda^A = X \ker p_\lambda$, as noted after Lemma 3.7, the submodule $\ker p_\lambda^A$ of X remains invariant under the action of $\varphi_X(A)$. Therefore, we can consider a linear map $\varphi_{X_\lambda} : A_\lambda \rightarrow L_{A_\lambda}(X_\lambda)$ defined by

$$\varphi_{X_\lambda}(\pi_\lambda^A(a))(\sigma_\lambda^X(x)) = \sigma_\lambda^X(\varphi_X(a)x), \forall a \in A, x \in X.$$

It is easy to check that $(\varphi_{X_\lambda})_\lambda$ is an inverse system of C^* -morphisms, such that $\varphi_X = \lim_{\leftarrow \lambda} \varphi_{X_\lambda}$, and thus (X, A, φ_X) is an inverse limit pro- C^* -correspondence. \square

4 Representations of pro- C^* -correspondences

Definition 4.1. *A morphism from a pro- C^* -correspondence (X, A, φ_X) to a pro- C^* -correspondence (Y, B, φ_Y) is a pair (Π, T) consisting of a pro- C^* -morphism $\Pi : A \rightarrow B$ and a map $T : X \rightarrow Y$ such that the following conditions are met:*

- (1) $\langle T(x_1), T(x_2) \rangle_B = \Pi(\langle x_1, x_2 \rangle_A)$, for all $x_1, x_2 \in A$;
- (2) $\varphi_Y(\Pi(a))T(x) = T(\varphi_X(a)x)$, for all $a \in A$ and for all $x \in X$.

We say that the morphism (Π, T) is *nondegenerate* if $[\Pi(A)B] = B$ and $[T(X)B] = Y$.

Remark 4.2. Let (Π, T) be a morphism from a pro- C^* -correspondence (X, A, φ_X) to a pro- C^* -correspondence (Y, B, φ_Y) . Then:

- (1) T is a continuous linear map.
- (2) $T(x) \Pi(a) = T(xa)$, for all $a \in A$, $x \in X$.

Proof. (1) A simple calculation, based on relation (1) of Definition 4.1, shows that T is linear.

For each $\delta \in \Delta$, there is $\lambda \in \Lambda$ such that

$$q_\delta^B (T(x))^2 = q_\delta (\Pi(\langle x, x \rangle_A)) \leq p_\lambda (\langle x, x \rangle_A) = p_\lambda^A (x)^2$$

for all $x \in X$.

(2) For each $\delta \in \Delta$, we have

$$\begin{aligned} & q_\delta^B (T(x) \Pi(a) - T(xa))^2 \\ &= q_\delta (\langle T(x) \Pi(a) - T(xa), T(x) \Pi(a) - T(xa) \rangle) \\ &= q_\delta (\Pi(a^* \langle x, x \rangle_A a) - \Pi(a^* \langle x, xa \rangle_A) - \Pi(\langle xa, x \rangle_A a) + \Pi(\langle xa, xa \rangle_A)) = 0, \end{aligned}$$

for all $a \in A$, $x \in X$. □

For the proof of Lemma 4.4, we use the following result from [KPW].

Lemma 4.3. ([KPW, Lemma 2.2]) *If A is a C^* -algebra and X is a Hilbert A -module, then for $n \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_n \in X$ we get that*

$$\left\| \sum_{i=1}^n \theta_{x_i, y_i} \right\| = \left\| ([\langle x_i, x_j \rangle_A]_{i,j=1}^n)^{\frac{1}{2}} ([\langle y_i, y_j \rangle_A]_{i,j=1}^n)^{\frac{1}{2}} \right\|,$$

where the norm in the right hand side is the norm in the C^* -algebra $M_n(A)$, of all $n \times n$ matrices with entries from A .

Lemma 4.4. *For a representation (Π, T) from a pro- C^* -correspondence (X, A, φ_X) to a pro- C^* -correspondence (Y, B, φ_Y) , there is a pro- C^* -morphism $\psi_T : K_A(X) \rightarrow K_B(Y)$, such that $\psi_T(\theta_{x,y}) = \theta_{T(x), T(y)}$, for all $x, y \in X$.*

Proof. It suffices to show that $\psi_T|_{\Theta(X)}$ is continuous. Since Π is continuous, for each $\delta \in \Delta$, there is $\lambda \in \Lambda$, such that $q_\delta (\Pi(a)) \leq p_\lambda(a)$, for all $a \in A$, and so there is a C^* -morphism $\Pi_\delta : A_\lambda \rightarrow B_\delta$ such that $\pi_\delta^B \circ \Pi = \Pi_\delta \circ \pi_\lambda^A$. Then for

each $\delta \in \Delta$, we have

$$\begin{aligned}
& q_{\delta, L_B(Y)}(\psi_T(\sum_{j=1}^n \theta_{x_j, y_j})) \\
&= q_{\delta, L_B(Y)}(\sum_{j=1}^n \theta_{T(x_j), T(y_j)}) = \|\sum_{j=1}^n \theta_{\sigma_\delta^Y(T(x_j)), \sigma_\delta^Y(T(y_j))}\| \\
&= \|([\pi_\delta^B(\langle T(x_i), T(x_j) \rangle_B)]_{i,j=1}^n)^{\frac{1}{2}}([\pi_\delta^B(\langle T(y_i), T(y_j) \rangle_B)]_{i,j=1}^n)^{\frac{1}{2}}\| \\
&= \|([\pi_\delta^B \circ \Pi(\langle x_i, x_j \rangle_A)]_{i,j=1}^n)^{\frac{1}{2}}([\pi_\delta^B \circ \Pi(\langle y_i, y_j \rangle_A)]_{i,j=1}^n)^{\frac{1}{2}}\| \\
&= \|([\Pi_\delta \circ \pi_\lambda^A(\langle x_i, x_j \rangle_A)]_{i,j=1}^n)^{\frac{1}{2}}([\Pi_\delta \circ \pi_\lambda^A(\langle y_i, y_j \rangle_A)]_{i,j=1}^n)^{\frac{1}{2}}\| \\
&= \|([\Pi_\delta(\langle \sigma_\lambda^X(x_i), \sigma_\lambda^X(x_j) \rangle_A)]_{i,j=1}^n)^{\frac{1}{2}}([\Pi_\delta(\langle \sigma_\lambda^X(y_i), \sigma_\lambda^X(y_j) \rangle_A)]_{i,j=1}^n)^{\frac{1}{2}}\| \\
&\leq \|([\langle \sigma_\lambda^X(x_i), \sigma_\lambda^X(x_j) \rangle]_{i,j=1}^n)^{\frac{1}{2}}([\langle \sigma_\lambda^X(y_i), \sigma_\lambda^X(y_j) \rangle]_{i,j=1}^n)^{\frac{1}{2}}\| \\
&= \|\sum_{j=1}^n \theta_{\sigma_\lambda^X(x_j), \sigma_\lambda^X(y_j)}\| = p_{\lambda, L_A(X)}\left(\sum_{j=1}^n \theta_{x_j, y_j}\right),
\end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $n \in \mathbb{N}$. \square

Let (X, A, φ_X) be a pro- C^* -correspondence. For each $\lambda \in \Lambda$, we define the ideals

$$J_X^\lambda = \{a \in A : \varphi_X^{L_A(X)}(a) \in K_{A_\lambda}(X_\lambda) \text{ and } \pi_\lambda^A(ab) = 0, \forall b \in \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)\}$$

and

$$J_X = \bigcap_{\lambda} J_X^\lambda.$$

Remark 4.5. For a C^* -correspondence (X, A, φ_X) , $J_X = \varphi_X^{-1}(K_A(X)) \cap (\ker \varphi_X)^\perp$ [K3, Definition 3.3] is the largest ideal to which the restriction of φ_X is an injection into $K_A(X)$. If (X, A, φ_X) is a C^* -correspondence, then

$$\begin{aligned}
J_X &= \{a \in A : \varphi_X(a) \in K_A(X) \text{ and } ab = 0, \forall b \in \ker \varphi_X\} \\
&= \varphi_X^{-1}(K_A(X)) \cap (\ker \varphi_X)^\perp = J_X.
\end{aligned}$$

Lemma 4.6. Let (X, A, φ_X) be an inverse limit pro- C^* -correspondence. Then $\pi_\lambda^A(J_X^\lambda) = J_{X_\lambda}$ for all $\lambda \in \Lambda$.

Proof. If (X, A, φ_X) is an inverse limit correspondence, then $\varphi_X = \lim_{\leftarrow \lambda} \varphi_{X_\lambda}$ and

$$\pi_\lambda^{L_A(X)} \circ \varphi_X = \varphi_{X_\lambda} \circ \pi_\lambda^A, \text{ for all } \lambda \in \Lambda. \text{ Therefore,}$$

$$\begin{aligned}
& \pi_\lambda^A(J_X^\lambda) \\
&= \{\pi_\lambda^A(a) \in A_\lambda : \varphi_{X_\lambda}(\pi_\lambda^A(a)) \in K_{A_\lambda}(X_\lambda), \pi_\lambda^A(a)\pi_\lambda^A(b) = 0, \forall b \in \ker(\varphi_{X_\lambda} \circ \pi_\lambda^A)\} \\
&= \{\pi_\lambda^A(a) \in A_\lambda : \varphi_{X_\lambda}(\pi_\lambda^A(a)) \in K_{A_\lambda}(X_\lambda), \pi_\lambda^A(a)\pi_\lambda^A(b) = 0, \forall \pi_\lambda^A(b) \in \ker \varphi_{X_\lambda}\} \\
&= J_{X_\lambda}
\end{aligned}$$

for all $\lambda \in \Lambda$. \square

- Definition 4.7.** (1) A representation of a pro- C^* -correspondence (X, A, φ_X) on a pro- C^* -algebra B is a morphism (π, t) from (X, A, φ_X) to the identity correspondence (B, B, id_B) .
- (2) A covariant representation of a pro- C^* -correspondence (X, A, φ_X) on a pro- C^* -algebra B is a representation (π, t) with the property that $\psi_t(\varphi_X(a)) = \pi(a)$, for all $a \in \mathcal{J}_X$.

Remark that in case (π, t) is a morphism of a pro- C^* -correspondence (X, A, φ_X) on a pro- C^* -algebra B , then the map $\psi_t : K_A(X) \rightarrow B$ of Lemma 4.4 is given by $\psi_t(\theta_{x,y}) = t(x)t(y)^*$, for $x, y \in X$. This is a consequence of Proposition 6.3 below and the fact that every pro- C^* -algebra has an approximate identity (see [F]).

5 Pro- C^* -algebras associated to pro- C^* -correspondences

For a representation (π, t) of a pro- C^* -correspondence (X, A, φ_X) on a pro- C^* -algebra B , we denote by $\text{pro-}C^*(\pi(A), t(X))$ the pro- C^* -subalgebra of B generated by the images of π and t .

Definition 5.1. For a pro- C^* -correspondence (X, A, φ_X) , the pro- C^* -algebra \mathcal{O}_X is defined to be the pro- C^* -algebra $\text{pro-}C^*(\pi_X(A), t_X(X))$, where (π_X, t_X) is a universal covariant representation of X , in the sense that for every covariant representation (π, t) of X on a pro- C^* -algebra B , there is a unique pro- C^* -morphism $\Phi : \mathcal{O}_X \rightarrow B$, such that $\Phi \circ \pi_X = \pi$, $\Phi \circ t_X = t$.

Remark 5.2. (1) If (X, A, φ_X) is a C^* -correspondence, then \mathcal{O}_X is the C^* -algebra associated to it [K1, Definition 2.6].

(2) Let (X, A, φ_X) be a pro- C^* -correspondence. If the pro- C^* -algebra \mathcal{O}_X exists, it is unique, up to a pro- C^* -isomorphism.

Lemma 5.3. Let (X, A, φ_X) be an inverse limit pro- C^* -correspondence with the property that $\pi_{\lambda\mu}^A(J_{X_\lambda}) \subset J_{X_\mu}$, for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. Then for each $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, there is a C^* -morphism $\rho_{\lambda\mu} : \mathcal{O}_{X_\lambda} \rightarrow \mathcal{O}_{X_\mu}$ such that $\rho_{\lambda\mu} \circ t_{X_\lambda} = t_{X_\mu} \circ \sigma_{\lambda\mu}^X$ and $\rho_{\lambda\mu} \circ \pi_{X_\lambda} = \pi_{X_\mu} \circ \pi_{\lambda\mu}^A$, where $(\pi_{X_\lambda}, t_{X_\lambda})$ is the universal covariant representation of Definition 5.1. Moreover, $\{\mathcal{O}_{X_\lambda}; \rho_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of C^* -algebras.

Proof. We easily get that for all $\lambda \geq \mu$, the pair $(\pi_{X_\mu} \circ \pi_{\lambda\mu}^A, t_{X_\mu} \circ \sigma_{\lambda\mu}^X)$ is a representation of the C^* -correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^* -algebra \mathcal{O}_{X_μ} . We will show that this representation is also a covariant representation. From

$$\begin{aligned} \psi_{t_{X_\mu} \circ \sigma_{\lambda\mu}^X}(\theta_{\sigma_\lambda^X(x), \sigma_\lambda^X(y)}) &= t_{X_\mu} \circ \sigma_{\lambda\mu}^X(\sigma_\lambda^X(x)) (t_{X_\mu} \circ \sigma_{\lambda\mu}^X(\sigma_\lambda^X(y)))^* \\ &= t_{X_\mu}(\sigma_\mu^X(x)) t_{X_\mu}(\sigma_\mu^X(y))^* \\ &= \psi_{t_{X_\mu}}(\theta_{\sigma_\mu^X(x), \sigma_\mu^X(y)}) = \psi_{t_{X_\mu}}\left(\pi_{\lambda\mu}^{L_A(X)}\left(\theta_{\sigma_\lambda^X(x), \sigma_\lambda^X(y)}\right)\right), \end{aligned}$$

for all $x, y \in X$, and taking into account that for all $\lambda \in \Lambda$, $\Theta(X_\lambda)$ is dense in $K_{A_\lambda}(X_\lambda)$, we deduce that $\psi_{t_{X_\mu} \circ \sigma_{\lambda\mu}^X} = \psi_{t_{X_\mu}} \circ \pi_{\lambda\mu}^{L_A(X)}|_{K_{A_\lambda}(X_\lambda)}$.

Let $\pi_\lambda^A(a) \in J_{X_\lambda}$, $a \in A$. Since $\pi_\mu^A(a) = \pi_{\lambda\mu}^A(\pi_\lambda^A(a)) \in J_{X_\mu}$, we have

$$\begin{aligned} \psi_{t_{X_\mu} \circ \sigma_{\lambda\mu}^X}(\varphi_{X_\lambda}(\pi_\lambda^A(a))) &= \psi_{t_{X_\mu}}\left(\pi_{\lambda\mu}^{L_A(X)}(\varphi_{X_\lambda}(\pi_\lambda^A(a)))\right) \\ &= \psi_{t_{X_\mu}}\left(\varphi_{X_\mu}(\pi_{\lambda\mu}^A(\pi_\lambda^A(a)))\right) \\ &= \psi_{t_{X_\mu}}\left(\varphi_{X_\mu}(\pi_\mu^A(a))\right) = \pi_{X_\mu}(\pi_\mu^A(a)) \\ &= \pi_{X_\mu} \circ \pi_{\lambda\mu}^A(\pi_\lambda^A(a)). \end{aligned}$$

Therefore, the pair $(\pi_{X_\mu} \circ \pi_{\lambda\mu}^A, t_{X_\mu} \circ \sigma_{\lambda\mu}^X)$ is a covariant representation of the C^* -correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^* -algebra \mathcal{O}_{X_μ} . From the universality of the covariant representation $(\pi_{X_\lambda}, t_{X_\lambda})$, there exists a unique C^* -morphism $\rho_{\lambda\mu} : \mathcal{O}_{X_\lambda} \rightarrow \mathcal{O}_{X_\mu}$, such that $\rho_{\lambda\mu} \circ t_{X_\lambda} = t_{X_\mu} \circ \sigma_{\lambda\mu}^X$ and $\rho_{\lambda\mu} \circ \pi_{X_\lambda} = \pi_{X_\mu} \circ \pi_{\lambda\mu}^A$. It is easy to check that $\{\mathcal{O}_{X_\lambda}; \rho_{\lambda\mu}; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of C^* -algebras. \square

Using Lemma 5.3 and following the proof of [JZ, Proposition 3.5], we obtain the following result, which gives a condition under which one has a covariant representation of an inverse limit pro- C^* -correspondence (X, A, φ_X) .

Proposition 5.4. *Let (X, A, φ_X) be an inverse limit pro- C^* -correspondence with the property that $\pi_{\lambda\mu}^A(J_{X_\lambda}) \subset J_{X_\mu}$, for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. Then there is a covariant representation (π, t) of (X, A, φ_X) on $\lim_{\leftarrow \lambda} \mathcal{O}_{X_\lambda}$.*

Proof. By Lemma 5.3, there is a pro- C^* -morphism $\pi = \lim_{\leftarrow \lambda} \pi_{X_\lambda}$ from A to $\lim_{\leftarrow \lambda} \mathcal{O}_{X_\lambda}$ and a map $t = \lim_{\leftarrow \lambda} t_{X_\lambda}$ from X to $\lim_{\leftarrow \lambda} \mathcal{O}_{X_\lambda}$. Following the proof of [JZ, Proposition 3.5], we show that (π, t) is a representation of (X, A, φ_X) on $\lim_{\leftarrow \lambda} \mathcal{O}_{X_\lambda}$. It is easy to check that $\psi_t = \lim_{\leftarrow \lambda} \psi_{t_{X_\lambda}}$. Let $a \in \mathcal{J}_X$. Then

$$\psi_t(\varphi_X(a)) = \left(\psi_{t_{X_\lambda}}(\varphi_{X_\lambda}(\pi_\lambda^A(a)))\right)_\lambda = (\pi_{X_\lambda}(\pi_\lambda^A(a)))_\lambda = \pi(a).$$

Therefore, (π, t) is a covariant representation of (X, A, φ_X) on $\lim_{\leftarrow \lambda} \mathcal{O}_{X_\lambda}$. \square

Next we find out an equivalent form of the condition $\pi_{\lambda\mu}^A(J_{X_\lambda}) \subset J_{X_\mu}$ in Proposition 5.4.

Lemma 5.5. *Let (X, A, φ_X) be an inverse limit pro- C^* -correspondence. Then the following statements are equivalent*

- (1) $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu)) = \{0\}$, for all $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$;
- (2) $\pi_{\lambda\mu}^A(J_{X_\lambda}) \subset J_{X_\mu}$, for all $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$.

Proof. If (X, A, φ_X) is an inverse limit pro- C^* -correspondence, then $\varphi_X = \lim_{\leftarrow \lambda} \varphi_{X_\lambda}$ and $\pi_\lambda^{L_A(X)} \circ \varphi_X = \varphi_{X_\lambda} \circ \pi_\lambda^A$, for all $\lambda \in \Lambda$.

(1) \Rightarrow (2) Let $\pi_\lambda^A(a) \in J_{X_\lambda}$, $a \in A$. Then

$$\varphi_{X_\mu}(\pi_\mu^A(a)) = \pi_{\lambda\mu}^{L_A(X)}(\varphi_{X_\lambda}(\pi_\lambda^A(a))) \in \pi_{\lambda\mu}^{L_A(X)}(K_{A_\lambda}(X_\lambda)) = K_{A_\mu}(X_\mu).$$

If $\pi_\mu^A(b) \in \ker \varphi_{X_\mu}$, $b \in A$, then

$$b \in \ker(\varphi_{X_\mu} \circ \pi_\mu^A) = \ker(\pi_\mu^{L_A(X)} \circ \varphi_X) = X^{-1}(\ker p_\mu).$$

Therefore,

$$\pi_\mu^A(a) \pi_\mu^A(b) = \pi_\mu^A(ab) \in \pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu)) = \{0\},$$

and so $\pi_\mu^A(a) = \pi_{\lambda\mu}^A(\pi_\lambda^A(a)) \in J_{X_\mu}$.

(2) \Rightarrow (1) Let $\pi_\mu^A(a) \in \pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu))$, $a \in A$. Since

$$\pi_\mu^A(J_X^\lambda) = \pi_{\lambda\mu}^A(\pi_\lambda^A(J_X^\lambda)) = \pi_{\lambda\mu}^A(J_{X_\lambda}) \subseteq J_{X_\mu},$$

where the second equality is due to Lemma 4.6, and since $\pi_\mu^A(X^{-1}(\ker p_\mu)) = \ker \varphi_{X_\mu}$ (see Remark 3.8) we have $\pi_\mu^A(a) \in J_{X_\mu} \cap \pi_\mu^A(X^{-1}(\ker p_\mu))$ and so $\pi_\mu^A(a) = 0$. \square

Remark 5.6. Since $\pi_\mu^A(\ker p_\mu) = \{0\}$ for all $\mu \in \Lambda$, $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu)) \subset \pi_\mu^A(\ker p_\mu)$ for all $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$ if and only if $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(\ker p_\mu)) = \{0\}$, for all $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$.

Remark 5.7. According to [K3, Definition 4.8] the condition $X(\ker p_\lambda) \subset \ker p_\lambda$ set out in Lemma 3.9 can be rephrased as $\ker p_\lambda$ is positively invariant for every $\lambda \in \Lambda$. Also the condition $\pi_{\lambda\mu}^A(J_{X_\lambda}) \subset J_{X_\mu}$ for all $\lambda, \mu \in \Lambda$, with $\lambda \geq \mu$, set out in Lemma 5.3 resembles to the notion of negative invariance of an ideal given in [K3, Definition 4.8].

Definition 5.8. Let (X, A, φ_X) be a pro- C^* -correspondence. An ideal I of A is *positively invariant* if $X(I) \subset I$, *negatively invariant* if $\pi_\mu^A(J_X^\lambda) \cap \pi_\mu^A(X^{-1}(I)) \subset \pi_\mu^A(I)$, for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$ and *invariant* if I is both positively and negatively invariant.

According to Definition 5.8, Lemma 3.9, Lemma 5.3, and Proposition 5.4 we get the following result.

Proposition 5.9. Let (X, A, φ_X) be a pro- C^* -correspondence. If $\ker p_\lambda, \lambda \in \Lambda$, are invariant, then there exists a covariant representation of (X, A, φ_X) on $\lim_{\leftarrow \lambda} \mathcal{O}_{X_\lambda}$.

In order to show in Theorem 5.10 below that \mathcal{O}_X exists, in case X is a pro- C^* -correspondence endowed with the property which is described in Proposition 5.9, we are going to use the notion of a \mathcal{T} -pair for a C^* -correspondence, which was introduced and studied in [K3, Sections 5-7]. We recall that given a C^* -correspondence (X, A, φ_X) , a \mathcal{T} -pair of X is a pair $\omega = (I, I')$ of ideals I, I' of A such that $X(I) \subset I$ and $I \subset I' \subset J(I) = \{a \in A : \phi_I(\varphi_X(a)) \in K_A(X/XI), aX^{-1}(I) \subset I\}$ [K3, Definition 5.6] (for the definition of ϕ_I see Remark 3.8). Also for two \mathcal{T} -pairs $\omega_1 = (I_1, I'_1)$, $\omega_2 = (I_2, I'_2)$, we denote $\omega_1 \subset \omega_2$, if $I_1 \subset I_2$ and $I'_1 \subset I'_2$ [K3, Definition 5.7].

Let (X, A, φ_X) be a pro- C^* -correspondence such that $\ker p_\lambda, \lambda \in \Lambda$ are invariant. For each $\lambda \in \Lambda$, $\omega_\lambda = (\{0\}, (\mathcal{J}_X)_\lambda)$ is a \mathcal{T} -pair of the C^* -correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$, since

$$(\mathcal{J}_X)_\lambda = \pi_\lambda^A(\mathcal{J}_X) \subset \pi_\lambda^A(J_X^\lambda) = J_{X_\lambda} = J(\{0\}).$$

Let $(\pi_{\omega_\lambda}, t_{\omega_\lambda})$ be the representation of the C^* -correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^* -algebra $\mathcal{O}_{X_{\omega_\lambda}}$ associated to the \mathcal{T} -pair ω_λ (see [K3, Definition 6.10]). Moreover, $\mathcal{O}_{X_{\omega_\lambda}}$ is generated by the images of t_{ω_λ} and π_{ω_λ} [K3, Proposition 6.11].

Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. Then $(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A, t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X)$ is a representation of the C^* -correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$, and let $\omega_{(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A, t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X)}$ be the \mathcal{T} -pair associated to this representation [K3, Definition 5.9]. Then, by definition,

$$\omega_{(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A, t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X)} = \left(\ker(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A), (\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A)^{-1} \left(\psi_{t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X}(K_{A_\lambda}(X_\lambda)) \right) \right).$$

Clearly $\{0\} \subset \ker(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A)$, and since

$$(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A)((\mathcal{J}_X)_\lambda) = \pi_{\omega_\mu}((\mathcal{J}_X)_\mu) \subset \psi_{t_{\omega_\mu}}(K_{A_\mu}(X_\mu)) = \psi_{t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X}(K_{A_\lambda}(X_\lambda)),$$

we have $\omega_\lambda \subset \omega_{(\pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A, t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X)}$. On the other hand,

$$C^*-(t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X(X_\lambda), \pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A(A_\lambda)) = C^*-(t_{\omega_\mu}(X_\mu), \pi_{\omega_\mu}(A_\mu)),$$

and then, by [K3, Theorem 7.1], there exists a unique surjective C^* -morphism $\rho_{\lambda\mu}^\omega : \mathcal{O}_{X_{\omega_\lambda}} \rightarrow \mathcal{O}_{X_{\omega_\mu}}$ such that $\rho_{\lambda\mu}^\omega \circ t_{\omega_\lambda} = t_{\omega_\mu} \circ \sigma_{\lambda\mu}^X$ and $\rho_{\lambda\mu}^\omega \circ \pi_{\omega_\lambda} = \pi_{\omega_\mu} \circ \pi_{\lambda\mu}^A$. It is easy to check that $\{\mathcal{O}_{X_{\omega_\lambda}}; \rho_{\lambda\mu}^\omega; \lambda, \mu \in \Lambda, \lambda \geq \mu\}$ is an inverse system of C^* -algebras.

The following theorem gives a condition under which \mathcal{O}_X exists.

Theorem 5.10. *Let (X, A, φ_X) be a pro- C^* -correspondence such that $\ker p_\lambda, \lambda \in \Lambda$, are invariant. Then there exists \mathcal{O}_X . Moreover, $\mathcal{O}_X = \lim_{\leftarrow \lambda} \mathcal{O}_{X_{\omega_\lambda}}$, up to a pro- C^* -isomorphism.*

Proof. By the above comments $(\pi_{\omega_\lambda})_\lambda$ is an inverse system of C^* -morphisms and $(t_{\omega_\lambda})_\lambda$ is an inverse system of linear maps. Let $t_\omega = \lim_{\leftarrow \lambda} t_{\omega_\lambda}$ and $\pi_\omega =$

$\lim_{\leftarrow \lambda} \pi_{\omega_\lambda}$. Following the proof of [JZ, Proposition 3.5], we show that (π_ω, t_ω) is a representation of (X, A, φ_X) on $\lim_{\leftarrow \lambda} \mathcal{O}_{X_{\omega_\lambda}}$. It is easy to check that $\psi_{t_\omega} = \lim_{\leftarrow \lambda} \psi_{t_{\omega_\lambda}}$. For $a \in \mathcal{J}_X$, we have

$$\begin{aligned} \psi_{t_\omega}(\varphi_X(a)) &= \left(\psi_{t_{\omega_\lambda}}(\varphi_{X_\lambda}(\pi_\lambda^A(a))) \right)_\lambda \\ &\quad [\text{K3, Lemma 5.10 (v)}] \\ &= \left(\pi_{t_{\omega_\lambda}}(\pi_\lambda^A(a)) \right)_\lambda = \pi_\omega(a). \end{aligned}$$

Therefore, (π_ω, t_ω) is a covariant representation of (X, A, φ_X) . Moreover, $\text{pro-}C^*(\pi_\omega(A), t_\omega(X)) = \lim_{\leftarrow \lambda} \mathcal{O}_{X_{\omega_\lambda}}$.

Let (π, t) be a covariant representation of (X, A, φ_X) on a pro- C^* -algebra B . Then, for each $\delta \in \Delta$, there exists a representation (π_δ, t_δ) of the C^* -correspondence $(X_\lambda, A_\lambda, \varphi_{X_\lambda})$ on the C^* -algebra B_δ such that $\pi_\delta^B \circ t = t_\delta \circ \sigma_\lambda^X$ and $\pi_\delta^B \circ \pi = \pi_\delta \circ \pi_\lambda^A$. Since,

$$\begin{aligned} \pi_\delta((\mathcal{J}_X)_\lambda) &= \pi_\delta^B \circ \pi(\mathcal{J}_X) = \pi_\delta^B(\psi_t(\varphi_X(\mathcal{J}_X))) \subset \pi_\delta^B(\psi_t(K_A(X))) \\ &= \psi_{t_\delta}(\pi_\lambda^{L_A(X)}(K_A(X))) = \psi_{t_\delta}(K_{A_\lambda}(X_\lambda)), \end{aligned}$$

$\omega_\lambda \subset \omega_{(t_\delta, \pi_\delta)}$, and then, by [K3, Theorem 7.1], there exists a surjective C^* -morphism $\tilde{\rho}_\delta : \mathcal{O}_{X_{\omega_\lambda}} \rightarrow C^*(t_\delta(X_\lambda), \pi_\delta(A_\lambda))$ such that $\tilde{\rho}_\delta \circ t_{\omega_\lambda} = t_\delta$ and $\tilde{\rho}_\delta \circ \pi_{\omega_\lambda} = \pi_\delta$. Therefore, there is a continuous $*$ -morphism $\rho_\delta : \lim_{\leftarrow \lambda} \mathcal{O}_{X_{\omega_\lambda}} \rightarrow B_\delta$, with $\rho_\delta = \tilde{\rho}_\delta \circ \chi_\lambda$, where χ_λ is the canonical projection from $\lim_{\leftarrow \lambda} \mathcal{O}_{X_{\omega_\lambda}}$ to $\mathcal{O}_{X_{\omega_\lambda}}$.

For each $\delta_1, \delta_2 \in \Delta$, such that $\delta_1 \geq \delta_2$, we have $\pi_{\delta_1 \delta_2}^B \circ \rho_{\delta_1} = \rho_{\delta_2}$ (see the proof of Proposition 3.5 [JZ]), and so there is a pro- C^* -morphism $\rho : \lim_{\leftarrow \lambda} \mathcal{O}_{X_{\omega_\lambda}} \rightarrow B$ such that $\pi_\delta^B \circ \rho = \rho_\delta$, for all $\delta \in \Delta$. It is easy to check that $\rho \circ t_\omega = t$ and $\rho \circ \pi_\omega = \pi$. Therefore the result follows from Definition 5.1 and Remark 5.2(2). \square

6 Pro- C^* -correspondences and crossed products of Hilbert pro- C^* -bimodules

Let X be a Hilbert bimodule over a pro- C^* -algebra A whose topology is given by the family of C^* seminorms $\{p_\lambda, \lambda \in \Lambda\}$.

Definition 6.1. [JZ, Definition 3.1] A *covariant representation* of a Hilbert $A - A$ pro- C^* -bimodule X on a pro- C^* -algebra B is a pair (φ_X, φ_A) consisting of a pro- C^* -morphism $\varphi_A : A \rightarrow B$ and a map $\varphi_X : X \rightarrow B$ which verifies the following relations:

- (1) $\varphi_X(xa) = \varphi_X(x)\varphi_A(a)$ and $\varphi_X(ax) = \varphi_A(a)\varphi_X(x)$ for all $x \in X$ and for all $a \in A$.
- (2) $\varphi_X(x)^*\varphi_X(y) = \varphi_A(\langle x, y \rangle_A)$ and $\varphi_X(x)\varphi_X(y)^* = \varphi_A(\langle x, y \rangle)$ for all $x, y \in X$.

Definition 6.2. [JZ, Definition 3.3] *The crossed product of A by X is a pro- C^* -algebra, denoted by $A \times_X \mathbb{Z}$, and a covariant representation (i_X, i_A) of (X, A) on $A \times_X \mathbb{Z}$ with the property that for any covariant representation (φ_X, φ_A) of (X, A) on a pro- C^* -algebra B , there is a unique pro- C^* -morphism $\Phi : A \times_X \mathbb{Z} \rightarrow B$ such that $\Phi \circ i_X = \varphi_X$ and $\Phi \circ i_A = \varphi_A$.*

We will show that the crossed product $A \times_X \mathbb{Z}$ of A by X is isomorphic to the pro- C^* -algebra \mathcal{O}_X associated to X , when X is regarded as a pro- C^* -correspondence.

The following result is a generalization of [Z, Theorem 6.5]. If X is a Hilbert $A - A$ pro- C^* -bimodule, then by ${}_A I$, we denote the closed ideal $\overline{\text{span}}\{{}_A < x, y > : x, y \in X\}$ of A .

Proposition 6.3. *Let X be a Hilbert $A - A$ pro- C^* -bimodule. Then ${}_A I = K_A(X)$, up to a pro- C^* -isomorphism.*

Proof. Since ${}_A I$ is a closed $*$ -ideal of A , it is a pro- C^* -algebra, hence we get that

$$\begin{aligned} {}_A I &= \lim_{\leftarrow \lambda} \overline{\pi_\lambda^A({}_A I)} = \lim_{\leftarrow \lambda} \overline{\pi_\lambda^A(\text{span}\{{}_A < x, y > : x, y \in X\})} \\ &= \lim_{\leftarrow \lambda} \overline{\text{span}\{{}_A < \sigma_\lambda^X(x), \sigma_\lambda^X(y) > : x, y \in X\}} \\ &= \lim_{\leftarrow \lambda} {}_{A_\lambda} I. \end{aligned}$$

From [BMS, Proposition 1.10], we have that for every $\lambda \in \Lambda$, there exists a C^* -isomorphism $\psi_\lambda : {}_{A_\lambda} I \rightarrow K_{A_\lambda}(X_\lambda)$ given by

$$\psi_\lambda(\pi_\lambda^A(a))(\sigma_\lambda^X(x)) = \pi_\lambda^A(a) \sigma_\lambda^X(x)$$

for all $a \in {}_A I$, $x \in X$. Moreover, for every $a \in {}_A I$, $x \in X$, $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$

$$\begin{aligned} ((\pi_{\lambda\mu}^{L_A(X)} \circ \psi_\lambda)(\pi_\lambda^A(a)))(\sigma_\mu^X(x)) &= \sigma_{\lambda\mu}^X(\psi_\lambda(\pi_\lambda^A(a))\sigma_\lambda^X(x)) \\ &= \sigma_{\lambda\mu}^X(\pi_\lambda^A(a)\sigma_\lambda^X(x)) = \sigma_\mu^X(ax) \\ &= \psi_\mu(\pi_\mu^A(a))(\sigma_\mu^X(x)) \\ &= (\psi_\mu \circ \pi_{\lambda\mu}^A)(\pi_\lambda^A(a))(\sigma_\mu^X(x)). \end{aligned}$$

Therefore $(\psi_\lambda)_{\lambda \in \Lambda}$ is an inverse system of C^* -isomorphisms between ${}_{A_\lambda} I$ and $K_{A_\lambda}(X_\lambda)$. Hence, since $K_A(X) = \lim_{\leftarrow \lambda} K_{A_\lambda}(X_\lambda)$, there is a unique pro- C^* -isomorphism $\psi : {}_A I \rightarrow K_A(X)$, such that $\psi(a)(x) = ax$ and $p_{\lambda, L_A(X)}(\psi(a)) = p_\lambda(a)$, for all $\lambda \in \Lambda$, $x \in X$, $a \in {}_A I$. \square

Proposition 6.4. *Let X be a Hilbert $A - A$ pro- C^* -bimodule. If X is viewed as a pro- C^* -correspondence over A , then $\mathcal{J}_X = {}_A I$.*

Proof. For each $\lambda \in \Lambda$, we have $\pi_\lambda^A(J_X^\lambda) = J_{X_\lambda} = {}_{A_\lambda} I = \pi_\lambda^A({}_A I)$ (for the equality $J_{X_\lambda} = {}_{A_\lambda} I$ see [K1, Lemma 2.4]). Then $a \in \mathcal{J}_X$ if and only if $\pi_\lambda^A(a) \in \pi_\lambda^A(J_X^\lambda) = \pi_\lambda^A({}_A I)$, for all $\lambda \in \Lambda$, that is if and only if $a \in {}_A I$. \square

Then from Proposition 6.4 and Proposition 6.3, we get the following corollary.

Corollary 6.5. *Let X be a Hilbert $A - A$ pro- C^* -bimodule. If X is viewed as a pro- C^* -correspondence over A , then $\mathcal{J}_X = K_A(X)$, up to a pro- C^* -isomorphism. Moreover, the pro- C^* -isomorphism from \mathcal{J}_X to $K_A(X)$ is given by $\Psi : \mathcal{J}_X \rightarrow K_A(X)$, $\Psi(a)x = ax$.*

Proposition 6.6. *Let (X, A, φ_X) be a pro- C^* -correspondence. Then the following assertions are equivalent:*

- (1) X has the structure of a Hilbert $A - A$ pro- C^* -bimodule;
- (2) $\varphi_X|_{\mathcal{J}_X}$ is a pro- C^* -isomorphism onto $K_A(X)$ such that $p_{\lambda, L_A(X)}(\varphi_X(a)) = p_\lambda(a)$, for all $a \in \mathcal{J}_X$, $\lambda \in \Lambda$.

Proof. (1) \Rightarrow (2) It follows from Corollary 6.5.

(2) \Rightarrow (1) It is easy to check that X has the structure of a left A -module with $ax = \varphi_X(a)(x)$, $a \in A, x \in X$ and ${}_A\langle x, y \rangle = (\varphi_X|_{\mathcal{J}_X})^{-1}(\theta_{x,y})$, $x, y \in X$, defines a left inner product on X . To show that X is a Hilbert $A - A$ bimodule, it remains to prove the coincidence of the topologies inherited on X by the two inner products. For all $x \in X$ and $\lambda \in \Lambda$, we have

$$\begin{aligned} {}^A p_\lambda(x)^2 &= p_\lambda({}_A\langle x, x \rangle) = p_\lambda((\varphi_X|_{\mathcal{J}_X})^{-1}(\theta_{x,x})) \\ &= p_{\lambda, L_A(X)}(\theta_{x,x}) = p_\lambda(\langle x, x \rangle_A) = p_\lambda^A(x)^2. \end{aligned}$$

□

Remark 6.7. In case (X, A, φ_X) is an inverse limit pro- C^* -correspondence and $\varphi_X|_{\mathcal{J}_X}$ is a pro- C^* -isomorphism onto $K_A(X)$, then $p_{\lambda, L_A(X)}(\phi_X(a)) = p_\lambda(a)$, for all $a \in \mathcal{J}_X$ and $\lambda \in \Lambda$. Indeed, since (X, A, φ_X) is an inverse limit pro- C^* -correspondence, $\varphi_X = \lim_{\leftarrow \lambda} \varphi_{X_\lambda}$, and it is easy to check that $\pi_\lambda^{L_A(X)} \circ \varphi_X|_{\mathcal{J}_X} = \varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda}$ for each $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. We will show that $\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda} : (\mathcal{J}_X)_\lambda \rightarrow K_{A_\lambda}(X_\lambda)$ is a C^* -isomorphism. Then it will follow that

$$\begin{aligned} p_{\lambda, L_A(X)}(\varphi_X(a)) &= \|\pi_\lambda^{L_A(X)}(\varphi_X(a))\| = \|\varphi_{X_\lambda}(\pi_\lambda^A(a))\| \\ &= \|\pi_\lambda^A(a)\| = p_\lambda(a) \end{aligned}$$

for all $a \in \mathcal{J}_X$. So, let $b \in \mathcal{J}_X$, such that $\varphi_{X_\lambda}(\pi_\lambda^A(b)) = 0$. Then $b \in \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)$ and therefore $b^* \in \ker(\pi_\lambda^{L_A(X)} \circ \varphi_X)$. Since $b \in \mathcal{J}_X$ we have $\pi_\lambda^A(b)\pi_\lambda^A(b^*) = 0$ and then $p_\lambda(b)^2 = p_\lambda(bb^*) = 0$. Therefore, $\pi_\lambda^A(b) = 0$ and thus $\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda}$ is injective. Furthermore $\varphi_{X_\lambda}|_{(\mathcal{J}_X)_\lambda}$ is surjective, since

$$\begin{aligned} \varphi_{X_\lambda}((\mathcal{J}_X)_\lambda) &= \varphi_{X_\lambda}(\pi_\lambda^A(\mathcal{J}_X)) = \pi_\lambda^{L_A(X)}(\varphi_X(\mathcal{J}_X)) \\ &= \pi_\lambda^{L_A(X)}(K_A(X)) = K_{A_\lambda}(X_\lambda). \end{aligned}$$

Remark 6.8. Let X be a Hilbert $A - A$ pro- C^* -bimodule. If X is regarded as a pro- C^* -correspondence, then, for each $\lambda \in \Lambda$, we have $(\mathcal{J}_X)_\lambda = \pi_\lambda^A(\mathcal{J}_X) = \pi_\lambda^A({}_A I) = {}_{A_\lambda} I = J_{X_\lambda}$.

Let $\lambda \in \Lambda$. Since $(\pi_{X_\lambda}, t_{X_\lambda})$ is an injective covariant representation of X_λ which admits a gauge action and

$$\omega_\lambda = (\{0\}, (\mathcal{J}_X)_\lambda) = (\{0\}, J_{X_\lambda}) = \omega_{(\pi_{X_\lambda}, t_{X_\lambda})},$$

by [K3, Theorem 7.1], there is a unique C^* -isomorphism $\rho_\lambda : \mathcal{O}_{X_{\omega_\lambda}} \rightarrow \mathcal{O}_{X_\lambda}$ such that $\rho_\lambda \circ t_{\omega_\lambda} = t_{X_\lambda}$ and $\rho_\lambda \circ \pi_{\omega_\lambda} = \pi_{X_\lambda}$.

On the other hand, by [K1, Proposition 3.7], \mathcal{O}_{X_λ} is canonically isomorphic to the crossed product $A_\lambda \times_{X_\lambda} \mathbb{Z}$ of A_λ by X_λ . Therefore, the C^* -algebras $\mathcal{O}_{X_{\omega_\lambda}}$ and $A_\lambda \times_{X_\lambda} \mathbb{Z}$ are canonically isomorphic.

Based on [JZ, Proposition 3.8], Remark 6.8 and Theorem 5.10 we have the following.

Proposition 6.9. *Let X be a Hilbert $A - A$ pro- C^* -bimodule. Then the pro- C^* -algebras \mathcal{O}_X and $A \times_X \mathbb{Z}$ are isomorphic, when X is regarded as a pro- C^* -correspondence.*

7 Pro- C^* -correspondences and pro- C^* -crossed products by automorphisms

Let A be a pro- C^* -algebra whose topology is given by the family of C^* -seminorms $\{p_\lambda; \lambda \in \Lambda\}$ and α a strong bounded automorphism of A (that is, for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that $p_\lambda(\alpha^n(a)) \leq p_\mu(a)$ for all $a \in A$ and for all integers n). We will show that the pro- C^* -algebra \mathcal{O}_A associated to the pro- C^* -correspondence (A, A, φ_A) (see Example 3.2) and $A \times_\alpha \mathbb{Z}$, the crossed product of A by α , are isomorphic as pro- C^* -algebras.

Indeed, if α is an automorphism of A as above, then (A, α, \mathbb{Z}) is a pro- C^* -dynamical system with the action of \mathbb{Z} on A given by $n \rightarrow \alpha^n$, and $A \times_\alpha \mathbb{Z}$ is the universal pro- C^* -algebra with respect to the nondegenerate covariant representations of (A, α, \mathbb{Z}) [J3, Definition 5.4 and Theorem 5.9].

If (u, φ) is a nondegenerate covariant representation of (A, α, \mathbb{Z}) on a pro- C^* -algebra B , then (π, t) , where $\pi = \varphi$ and $t(a) = u_1^* \varphi(a)$ is a nondegenerate representation of (A, A, φ_A) on B . Moreover, this representation is covariant. Indeed, since $K_A(A) = A$, the pro- C^* -morphism ψ_t is given by $\psi_t(a) = u_1^* \varphi(a) u_1$, and then

$$\begin{aligned} \psi_t(\varphi_A(a)) &= u_1^* \varphi(\varphi_A(a)) u_1 = u_1^* \varphi(\alpha(a)) u_1 \\ &= u_1^* u_1 \varphi(a) u_1^* u_1 = \varphi(a) = \pi(a), \end{aligned}$$

for all $a \in \mathcal{J}_A$.

Conversely, if (π, t) is a nondegenerate covariant representation of (A, A, φ_A) on a pro- C^* -algebra B , then the map $u : B \rightarrow B$ defined by $u(t(a)b) = \pi(a)b$

is a unitary operator, and (u, φ) , where $\varphi = \pi$ and $n \rightarrow u_n = u^n$ with $u_0 = \text{id}_B$, is a nondegenerate covariant representation of (A, α, \mathbb{Z}) on B .

We remark that if (π, t) is a covariant representation of a nondegenerate pro- C^* -correspondence (X, A, φ_X) on a pro- C^* -algebra B , then (π, t) is a nondegenerate covariant representation of (X, A, φ_X) on the pro- C^* -algebra $\text{pro-}C^*-\{t(X), \pi(A)\}$.

Using these facts and the universal property for crossed products of pro- C^* -algebras [J3, Corollary 5.7], we have the following proposition.

Proposition 7.1. *Let A be a pro- C^* -algebra, whose topology is given by the family of C^* -seminorms $\{p_\lambda; \lambda \in \Lambda\}$ and let α be an automorphism of A with the property that for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that $p_\lambda(\alpha^n(a)) \leq p_\mu(a)$, for all $a \in A$ and for all integers n . Then the pro- C^* -algebras \mathcal{O}_A and $A \rtimes_\alpha \mathbb{Z}$ are isomorphic.*

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